WHOSE LIMIT IS IT ANYWAY?

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Abstract. In a tongue-in-cheek manner, we investigate the notion of limit. We illustrate some of its shortcomings and show that addressing these shortcomings can often lead to unexpected consequences.

This article was inspired by the myriad answers, excuses, embarrassed looks and extended discussions I engendered when innocently asking many colleagues to compute $\lim_{x\to 0}\sqrt{x}$. Of course, the two obvious answers, "zero" and "does not exist", were eagerly proffered. I started then to contemplate why these professional mathematicians and educators (including myself) were in seeming disagreement over such an apparently simple question. After all, I had been telling students of mine for years that mathematics is a precise science. It is universal. A carefully posed purely mathematical question (the kind we hope we put on our exams) has an irrefutably accurate answer. In pursuit of the answer as to why mathematics had seemingly failed to give this "irrefutably accurate answer" to the problem of computing $\lim_{x\to 0}\sqrt{x}$, I embarked on a long, enlightening journey. Although the path the journey takes us is much too treacherous for first-year students, as it is a path ravaged by scoundrel functions, it is certainly one in which the seasoned mathematical adventurer will surely find challenge and delight. In a tongue-in-cheek style that I hope you, the reader, find enjoyable, I now recount that journey.

Many individuals who would consider themselves fluent in mathematics would no doubt agree that the concept of limit is fundamental. If asked to give a mathematically rigorous definition of the (2–sided) limit of a function $f(x)$, the most frequent response would most likely be a recitation of the mantra:

Limit Definition 1: $\lim\limits_{x\to a} f(x) = L$ means that given any $\varepsilon > 0$, there exists $\delta > 0$ so that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$.

What is interesting and often overlooked about this definition is that it is only valid for "simple" functions, that is those functions whose domains contain a deleted neighborhood of $x = a$. Simple functions include polynomial and rational functions, but certainly exclude many algebraic and transcendental functions.

As an example, consider the function (shown in Figure 1)

$$f(x) = \sqrt{x\sin(1/x)}$$

The domain of $f(x)$ does not contain any deleted open neighborhood of $x = 0$. The graph of $f(x)$ certainly suggests that $\lim\limits_{x\to 0} f(x) = 0$, so to prove $\lim\limits_{x\to 0}\sqrt{x\sin(1/x)} = 0$ by definition requires that given $\varepsilon > 0$ there is a $\delta > 0$ so that $|f(x)| < \varepsilon$ whenever $0 < |x| < \delta$. Of course, no such δ exists, as $f(x)$ is undefined for some values of x arbitrarily close to $x = 0$. One likely response is to conclude that the limit does not exist, but that seems wholly unsatisfactory since it goes against our intuition as supported by the graph of $f(x)$. In addition, later in this paper we will see that the function $f(x)$ is continuous at $x = 0$ (if we define $f(0) = 0$), whereby the conclusion that $\lim\limits_{x\to 0} f(x)$ does not exist becomes absurd.

When confronted with this particular conundrum, many might respond with an exasperated "It's the domain! You forgot to consider the domain," and offer another enticing and shrewd definition of limit:

Limit Definition 2: $\lim\limits_{x\to a} f(x) = L$ means that given any $\varepsilon > 0$, there exists $\delta > 0$ so that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$ and $x \in$ domain of $f(x)$.

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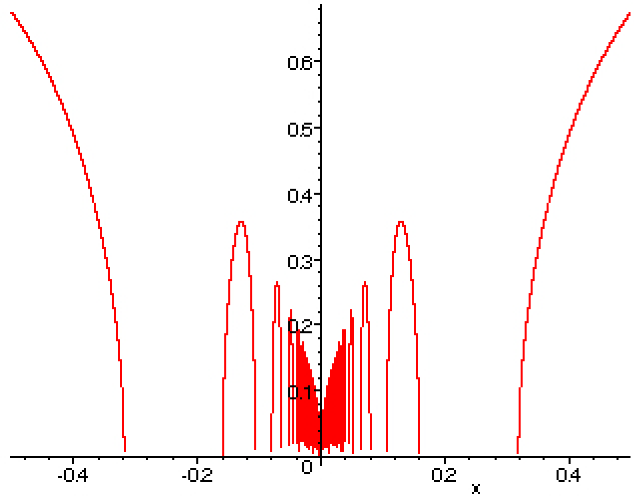

FIGURE 1. Graph of the function $f(x) = \sqrt{x \sin(1/x)}$

Smugly, we conclude that $\lim_{x\to 0} \sqrt{x\sin(1/x)} = 0$.

Finding an elementary text that includes the domain of a function $f(x)$ as part of its definition for limit is very difficult. See [1,5,9,16]. To their authors' credit, many books [2,3,6,10,11,12,17,18,20] do include the caveat that limit taking is only to be performed on functions defined on deleted neighborhoods of $x = a$. That, however, leaves our friend $f(x) = \sqrt{x\sin(1/x)}$ and its behavior near $x = 0$ too scurrilous for consideration. The only undergraduate texts I have found that can handle $\lim_{x\to 0} \sqrt{x\sin(1/x)}$ are [4,7,8,13,14,15]. Only the books [7] and [15] are intended as an introduction to calculus and [15] is out of print. But, even in the book of Courant and John [7], one is faced with a departure from the conventional notion of two–sided limit:

> **Courant–John Limit Definition (**[7, Section 1.8]**):** $\lim_{x\to a} f(x) = L$ means that whenever an arbitrary quantity ε is assigned we can mark off an interval $|x - a| < \delta$ so small that for any x which belongs both to the domain of f and to that interval the inequality $|f(x) - L| < \varepsilon$ holds.

Thus, if we let $p(x) \equiv 1$ for $x \neq 0$ and define $p(0) = 2$, and try to evaluate $\lim_{x\to 0} p(x)$ using the Courant–John definition we would conclude that the limit does not exist! This is because, according to their definition, the interval to be "marked off" must always contain the point $x = 0$.

Unfortunately, while we bask in the glory of our success, another rogue function $g(x) = (x^4 - x^2)^{3/2}$ intrudes. See Figure 2.

The domain of $g(x)$ is $(-\infty, -1] \cup \{0\} \cup [1, \infty)$. Note that in contrast to the domain of $f(x)$, $x = 0$ is an isolated point in the domain of $g(x)$. Armed with the power of our modified limit definition, we enter the fray and attempt a swift defeat of computing $\lim_{x\to 0} g(x)$. A reasonable guess is that $\lim_{x\to 0} g(x) = 0$, but upon brandishing our newly forged definition of limit, we find that we cannot even check the validity of the assertion that $\lim_{x\to 0} g(x) = 0$ since $\{x \mid 0 < |x| < \delta\} \cap \{\text{domain of } f(x)\} = \emptyset$ for δ small. For that matter, we may as well attempt to show that $\lim_{x\to 0} g(x) = \pi$. The only expedient retreat from this debacle is to further modify the definition of limit.

We need first the definition of an accumulation point. The point a is an *accumulation point* for a set S of real numbers if for each $\delta > 0$, there exists a point $s \in S$ with $0 < |s - a| < \delta$. A point of S that is not an accumulation point of S is called an *isolated point* of S.

> **Limit Definition 3:** Let a be an accumulation point of the domain of a function $f(x)$. $\lim_{x\to a} f(x) = L$ means that given any $\varepsilon > 0$, there exists $\delta > 0$ so that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$ and $x \in$ domain of $f(x)$.

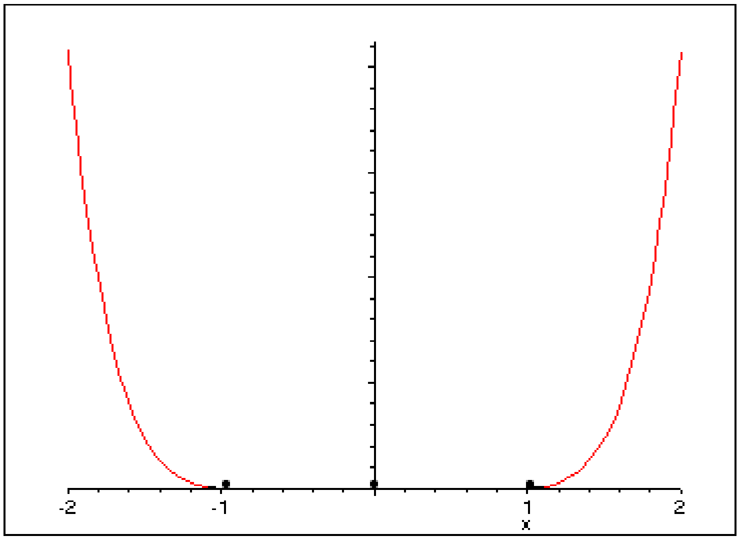

FIGURE 2. Graph of the rogue function $g(x) = (x^4 - x^2)^{3/2}$

Since $x = 0$ is not an accumulation point for the domain of $g(x)$, computation of $\lim_{x \to 0} g(x)$ is ill-posed. Thus, we reluctantly admit that evaluation of $\lim_{x \to 0} g(x)$ is not legitimate. In any case, there is still unrest in the streets because an aspiring mathematical acolyte has posted a bill in the town square that reads:

> Let $g(x) = (x^4 - x^2)^{3/2}$. Then a simple application of the Chain Rule yields $g'(x) = \frac{3}{2}(x^4 - x^2)^{1/2}(4x^3 - 2x)$. So $g'(0) = 0$ and hence $g(x)$ is differentiable at $x = 0$. One must then conclude that $g(x)$ is continuous at $x = 0$, and thus $\lim_{x \to 0} g(x) = g(0) = 0$.

To quell the unrest, we, the wise town elders, convene to draft our response to these questionable writings. Our strategy will be to denounce the validity of this particular application of the Chain Rule. First, we consult one of many revered tomes for the definition of differentiability. We find that a function $f(x)$ is differentiable at $x = c$ if the familiar limit of the difference quotient exists:

$$\lim_{h \to 0} \frac{f(c + h) - f(c)}{h}$$

In the case when this limit exists, we denote it by $f'(c)$. Unfortunately, if we apply limit definition 3 to compute the limit of the difference quotient we will soon find ourselves in unwelcome collusion with the acolyte, for we can validate his logic:

> The Chain Rule states that $h(x) = (f_1 \circ f_2)(x)$ is differentiable at $x = c$ if $f_2(x)$ is differentiable at $x = c$ and $f_1(x)$ is differentiable at $f_2(c)$. In this case, $h'(c) = f_1'(f_2(c)) \cdot f_2'(c)$. If we let $f_1(x) = x^{3/2}$ and $f_2(x) = x^4 - x^2$, then $g(x) = (x^4 - x^2)^{3/2} = (f_1 \circ f_2)(x)$. $f_2(x)$ certainly poses no differentiability problems, so what of the differentiability of $f_1(x)$ at $f_2(0) = 0$? The limit of the difference quotient yields $\lim_{h \to 0} \dfrac{h^{3/2}}{h} = 0$ by our enlightened limit definition 3. We are forced to now lie in agreement with the acolyte that $g'(0) = 0$, and since differentiability implies continuity, we must also agree with his questionable claim that $\lim_{x \to 0} g(x) = g(0) = 0$.

We have now shown that our modified definition of limit can lead to a contradiction of itself! On one hand, limit definition 3 implies $\lim_{x \to 0} g(x)$ is illegitimate, and on the other hand limit definition 3 implies existence of $g'(0)$, and thus the existence of $\lim_{x\to 0} g(x)$. We have surely come upon dark days! Decisive measures will be taken to rebuff the acolyte and his impious logic! In fact, we have shown that if we use limit definition 3, the composition of differentiable functions may not be differentiable! Careful analysis of the proof of the Chain Rule exposes the necessity that in order for $h(x)$ to be differentiable at $x = c$, the following additional technical condition must hold:

Condition (†) If $A_\delta = \{x \mid 0 < |x - c| < \delta \text{ and } x \in \text{ domain of } f_2(x)\}$ then $f_2(A_\delta) \cap \{\text{domain of } f_1(x)\} \neq \emptyset$ for all $\delta > 0$.

To see the necessity of condition (†), let's review the proof of the Chain Rule:

Suppose f_1 is differentiable at $u = f_2(c)$ and that f_2 is differentiable at c. Define the function

$$\Phi(k) = \begin{cases} \dfrac{f_1(u+k) - f_1(u)}{k} - f_1'(u) & \text{for } k \neq 0,\ u + k \in \text{ domain of } f_1(x) \\ 0 & \text{for } k = 0 \end{cases}$$

By definition of derivative, $\lim_{k\to 0} \Phi(k) = f_1'(u) - f_1'(u) = 0 = \Phi(0)$. Thus, $\Phi(k)$ is continuous at $k = 0$. Now, it is always true that $f_1(u+k) - f_1(u) = [f_1'(u) + \Phi(k)]\, k$. Let $u = f_2(c)$ and $k = f_2(c+h) - f_2(c)$. Thus, for $u + k \in$ domain of $f_1(x)$, which is the same as $f_2(c+h) \in$ domain of $f_1(x)$, we have

$$(*) \qquad f_1(f_2(c+h)) - f_1(f_2(c)) = [f_1'(f_2(c)) + \Phi(k)]\, [f_2(c+h) - f_2(c)]$$

Hence

$$(f_1 \circ f_2)'(c) = \lim_{h\to 0} \frac{f_1(f_2(c+h)) - f_1(f_2(c))}{h} = \lim_{h\to 0} [f_1'(f_2(c)) + \Phi(k)]\, \frac{f_2(c+h) - f_2(c)}{h}$$
$$= [f_1'(f_2(c)) + 0] f_2'(c) = f_1'(f_2(c)) f_2'(c)$$

The last line follows since $\lim_{h\to 0} \Phi(k) = \lim_{k\to 0} \Phi(k) = 0$, since f_2 is continuous at c because it is differentiable there. One can now easily see the necessity of condition (†), by considering under what circumstances (*) is a meaningful expression.

In the acolyte's example, $f_2(A_\delta) \cap \{\text{domain of } f_1(x) = x^{3/2}\} = \emptyset$ for $\delta < 1$, so the Chain Rule does not apply. In fact, direct application of the definition of derivative to $g(x)$ at $x = 0$ yields:

$$\lim_{h\to 0} \frac{(h^4 - h^2)^{3/2}}{h}$$

which is not a legitimate limit since the there are no small values of $h \neq 0$ for which the quotient is defined. We conclude with finality that $g(x) = (x^4 - x^2)^{3/2}$ is not differentiable at $x = 0$.

We instruct the scribes to incorporate these clarifications in all appropriate future mathematical volumes and send the town crier out to make the clarifying announcement and post a homework exercise to show that the function

$$f(x) = \begin{cases} \left[x^3 \sin(1/x)\right]^{3/2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

is differentiable at $x = 0$.

Although publicly humiliated, the acolyte vows that the town elders will someday pay for their reckless disregard for careful and accurate definition. To this end, he decides to venture out on a quest to consult with a master of higher mathematical arcana: a topologist. The topologist explains that it is in fact true that the function $g(x) = (x^4 - x^2)^{3/2}$ is continuous. However, he adds that the assertion $\lim_{x\to 0} g(x) = g(0) = 0$ is illegitimate. The acolyte, being uncertain of his ability to remember all of the clever complex twists of mathematical machination required to establish the continuity of $g(x)$, asks the topologist to prepare a manuscript detailing this formidable logic. The manuscript reads:

Let X and Y be two topological spaces. A function $f : X \longrightarrow Y$ is continuous if the inverse image of every open set is open. That is, f is continuous if for every open set U in Y, $f^{-1}(U)$ is an open set in X. In the case of $g(x)$, $X = (-\infty, -1] \cup \{0\} \cup [1, \infty)$, and $Y = [0, \infty)$ both with the induced topology from $\mathbb{R}$. It suffices to verify the continuity condition for basic open sets U in Y. A basic open set in Y is the intersection of some open interval of $\mathbb{R}$ with Y. Hence there are two types. One type looks like $U_1 = (a, b)$ with $0 < a < b \leq \infty$ and the other type is of the form $U_2 = [0, b)$, $0 < b \leq \infty$. It's easy to see, from the graph of $g(x)$, that $f^{-1}(U_1)$ is the union V_1 of two open intervals $(-\beta, -\alpha) \cup (\alpha, \beta)$, $1 < \alpha < \beta \leq \infty$. V_1 is clearly open in X. Similarly, $f^{-1}(U_2)$ is of the form $V_2 = (-\beta, -1] \cup \{0\} \cup [1, \beta)$, $1 < \beta \leq \infty$. Since $V_2 = (-\beta, \beta) \cap X$, V_2 is open in X. Thus $g(x)$ is continuous. However, since $x = 0$ is an isolated point in the domain of $g(x)$, the limit $\lim\limits_{x \to 0} g(x)$ is ill–posed.

Recall the conventional definition of continuity:

Continuity Definition 1: A function $f(x)$ is continuous at $x = x_0$ if $\lim\limits_{x \to x_0} f(x) = f(x_0)$.

The function $g(x)$ is not continuous at $x = 0$ by this definition, even using limit definition 3. After careful examination of the topologist's manuscript, we can only conclude that, in addition to our original limit definition 1, our definition of continuity is also inadequate. After meticulous discussion we settle on a definition of continuity consistent with the topologist's topological definition:

Continuity Definition 2: Let $f : X \subset \mathbb{R} \to \mathbb{R}$. Then f is continuous at $x_0 \in X$ if either x_0 is an isolated point of X, or $\lim\limits_{x \to x_0} f(x) = f(x_0)$.

Note that this definition of continuity also implies continuity of $f(x) = \sqrt{x \sin(1/x)}$ discussed at the beginning of this article if we define $f(0) = 0$.

And thus ends our journey. What a long, strange trip it's been! Of course, we would never subject first-year students to such limit and continuity esoterica, but the issues mentioned above can be the basis for an interesting discussion in introductory courses on real analysis. After all, this whole journey was inspired by the confusion among mathematical professionals over the correct answer to $\lim_{x \to 0} \sqrt{x}$.

Admittedly, we had to go out our way to come up with examples for which the conventional limit definition 1 failed to give satisfactory results that matched our graphical intuition. All of our "misbehaving" examples were of functions which were not defined on deleted neighborhoods. Why don't we avoid these issues entirely and restrict ourselves to functions that are only defined on deleted neighborhoods? The reason is that when one considers functions of several variables, this "simplification" requires that most rational functions must be discarded. As a simple example consider the function $f(x, y) = \dfrac{x^2y^2 - 1}{xy - 1}$. Would anyone like to refute that

$$\lim_{(x,y) \to (1,1)} \frac{x^2y^2 - 1}{xy - 1} = \lim_{(x,y) \to (1,1)} (xy + 1) = 2$$

and claim that this limit cannot be taken since $f(x, y)$ is certainly not defined on any deleted neighborhood of the point $(1, 1)$, and, as such, is a scoundrel of 2–dimensions, just as vulgar, but seemingly not as contrived, as its 1–dimensional cousins? Can one convincingly deny that

$$\lim_{(x,y) \to (0,0)} \frac{\sin y}{y} = 1 \ ?$$

These two limits are off-limits for the books [5,10,12,17] and some of the books even have exercises which are invalid given their definition of limit [12, Ex. 24,26 Section 12.2] and [5, Ex. 11,15 Section 12.2], for example. Curiously, the book [10, Ex. 11 Section 15.1] gives the exercise

$$\lim_{(x,y)\to(1,1)} \frac{x^2 - y^2}{x - y}$$

and states in the solutions at the back of the book that this limit does not exist: A consistent answer, given its definition of limit!

So who's limit is it anyway? It should be ours: the professional mathematicians and mathematics educators. Even though we will vigorously and authoritatively defend our proposed answers to $\lim_{x\to 0}\sqrt{x}$, do we dare risk asking whether or not we are defending the correct answer? We certainly ask our students to do this. I think the joke is on us!

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